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Colin Naturman

Abstract: Certain topological countability properties are generalized to interior algebras and basic results concerning these properies are investigated. The preservation of these properties under the formation of principal quotients and under a new construction called a *join* of interior algebras, is investigated.

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Introduction

The study of countability properties in topology deals with results arising when certain natural sets associated with a topological space can be chosen to be countable. The most interesting cases being when every base or neighbourhood base is countable (second and first countability) or when every open cover contains a countable cover (the Lindelöf property.) In this paper we generalize these properties to interior algebras and show that that the main results concerning these properties in topology still hold for interior algebras.

In §1 we briefly investigate bases for generalized topologies and interior algebras, the natural generalization of bases for topologies. Notions of bases and subbases for interior algebras were already present in the early literature on interior algebras in some form or another. (See [1] or page 95 of [4] for example.) We treat bases and subbases (Definitions 1.1 and 1.4) using a generalized topology approach. In §2 we generalize first countability, second countability and the Lindelöf property to interior algebras (Definitions 2.1, 2.2 and 2.3). We show that every second countable interior algebra is first countable and Lindelöf (Theorem 2.4). What is particularly interesting is that second countability still implies the Lindelöf property even though the standard proof of this result in topology does not generalize to interior algebras. (The standard proof makes use of points and the ability to take arbitrary unions which correspond respectively to atoms and arbitrary joins in an interior algebra, yet our result will hold in any interior algebra regardless of whether it has atoms or whether it is complete.) We also relate the Lindelöf property to the accumulation of countably complete filters (Proposition 2.6, Theorem 2.7). We introduce the concept of a *sequentially determined* interior algebra (Definition 2.9) which are interior algebras in which the canonical pre–order [2] is completely determined by sequence convergence. Every first countable interior algebra is sequentially determined (Theorem 2.10). The properties introduced in §2 are particularly interesting as regards their presevation under the formation of principal quotients and under a new construction known as a *join* of interior algebras. In §3 we investigate the preservation of the countability properties under the formation of principal quotients. The classes of first countable, second countable and sequentially determined interior algebras are all closed under principal quotients (Theorem 3.3). The preservation of second countability under principal quotients allows us to show that an interior algebra is second countable if and only if every base for it contains a countable base (Theorem 3.4). (This result is also interesting with regard to the fact that the proof of the corresponding result in topology does not generalize automatically to interior algebras.) The class of Lindelöf algebras are closed under closed quotients (Theorem 3.5). An interior algebra with the property that all its principal quotients are Lindelöf is said to be *hereditarily Lindelöf* (Definition 3.7). Every second countable interior algebra is hereditarily Lindelof (Theorem 3.10). In §4 we introduce *joins* of interior algebras (Definition 4.1) and investigate the preservation of the countability properties under joins. The classes of first countable and sequentially determined interior algebras are closed under open joins (Theorem 4.6), the classes of Lindelöf algebras and hereditarily Lindelöf algebras are closed under countable joins (Theorem 4.9), and the class of second countable interior algebras is closed under countable open joins (Theorem 4.10).

Notation and Terminology

Bold capitals will be used to denote structures. If **A**, **B**, **C**, ... are structures, then the corresponding normal capitals A, B, C, ... will denote the underlying sets of these structures. If f : **A** ⟶ **B** is a map between structures then ker(f) denotes the *kernel* of f, i.e the equivalence relation on A given by $(a,b) \in \ker(f)$ if and only if $f(a) = f(b)$.

For required results concerning interior algebras we refer the reader to [2] and [3] and the references given there. We briefly revise some important definitions and notational conventions. If **A** is an interior algebra then ^{I} and ^{C} will denote its interior operator and closure operator respectively. The Heyting algebra of open elements of **A** is denoted by $\mathbf{A}^O$

and the dual Heyting algebra of closed elements of A is denoted by $A^{\square}$. Elements which are both open and closed are called *clopen*. Ba A will denote the underlying Boolean algebra of A. The *neighbourhood function* [2] of A will be denoted by N, so that if $a \in A$, $N(a) = \{b \in A : a \leq b^{I}\}$, the set of *neighbourhoods* [2] of a. Consider a net (a_i) in A based on a directed set $W = \langle W, \sqsubseteq \rangle$. If $a_i > 0$ for all $i \in W$ then we say that (a_i) is *non–zero* [2]. We say that (a_i) *converges* [2] to $b \in A$ and that b is a *limit* [2] of (a_i), denoted by $(a_i) \longrightarrow b$, if and only if for all $d \in N(b)$ there is a $j \in W$ with $a_i \leq d$ for all $i \sqsupseteq j$ in W. We say that (a_i) *accumulates* [2] at b and that b is an *accumulant* [2] of (a_i), denoted by $(a_i) \longleftarrow b$, if and only if for all $d \in N(b)$ and all $j \in W$, there is an $i \sqsupseteq j$ in W with $a_i \leq d$. $\ll$ denotes the *canonical pre–order* [2] on A which is given by $a \ll b$ in A if and only if $a \leq b^{C}$. If $R \subseteq A$ then we say that R *accumulates* [2] at $b \in A$ and that b is an *accumulant* [2] of R, denoted by $R \longleftarrow b$, if and only if $b \ll r$ for all $r \in R$.

If A is an interior algebra and $c \in A$ then $(c]$ denotes the principal ideal $\{b \in A : b \leq c\}$. The *ideal algebra* of A generated by c is the interior algebra $(c] = \langle (c], \cdot, +, {}'^{c}, 0, a, {}^{I_c}, {}^{C_c} \rangle$ where for all $b \in (c]$ we have $b'^{c} = cb'$, $b^{I_c} = c \cdot (c' + b)^{I}$ and $b^{C_c} = cb^{C}$. We will denote the neighbourhood function of $(c]$ by N^{c}. (Ideal algebras are a special case of *interval algebras* for results about which we refer the reader to [3].)

If X is a subset of a Boolean algebra or interior algebra then ΣX and $\sqcap X$ denote the join and meet of X respectively. A filter F in a Boolean algebra or interior algebra is said to be *sequential* [2] if and only if it has a countable base. F is called (*countably*) *complete* if and only if for every (countable) subset $S \subseteq F$, if $\sqcap S$ exists then $\sqcap S \in F$. A homomorphism between Boolean algebras or between interior algebras is called *complete* if and only if it preserves all meets and joins.

If B is a Boolean algebra then a subset $G \subseteq B$ is called a *generalized topology* [2] in B if and

only if (i): $0,1 \in G$, (ii): G is closed under arbitrary joins, (iii): G is closed under finite meets, and (iv): for all $a \in B$, $\Sigma \{b \in G : b \leq a\}$ exists. (More simply, by Proposition 1.2 of [2], G is a generalized topology in B if and only if (i): $1 \in G$, (ii): G is closed under binary meets, and (iii): for all $a \in B$, $\max \{b \in G : b \leq a\}$ exists.) A structure $\langle B, G \rangle$ is called a *generalized topological space* [2] if and only if B is a Boolean algebra and G is a generalized topology in B. For basic results concerning generalized topological spaces we refer the reader to [2] and [3]. If $T = \langle B, G \rangle$ is a generalized topological space, Alg T denotes its corresponding interior algebra with interior operator given by $a^{I} = \max \{b \in G : b \leq a\}$ for all $a \in B$. If A is an interior algebra then Gt A denotes its corresponding generalized topological space $\langle \text{Ba } A, A^{O} \rangle$. Consider a topological space $X = \langle X, \mathcal{T} \rangle$ and let B be the power set Boolean algebra on X. Then $\mathcal{T}$ is a generalized topology in B. We will use $\mathcal{P}(X)$ to denote the corresponding interior algebra Alg $\langle B, \mathcal{T} \rangle$, called the *power set interior algebra determined by* X. If A and C are interior algebras and there is a congruence θ on Gt A (considered as a Boolean algebra with added unary relation) such that $C \cong \text{Alg (Gt } A)/\theta$, then we say that C is a *principal quotient* [3] of A. By Theorem 3.7 of [3] we have that C is a principal quotient of A if and only if there is a $c \in A$ with $C \cong (c]$. If we can choose c to be open we say that C is an *open quotient* [3] of A. Similarly we may define *closed quotients* and *clopen quotients* [3].

If A and C are interior algebras then a map $f : A \longrightarrow C$ is called a *topomorphism* [3] if and only if f is a Boolean algebra homomorphism and $f[A^{O}] \subseteq B^{O}$. (Notice that $f : A \longrightarrow B$ is a topomorphism if and only if $f :$ Gt $A \longrightarrow$ Gt C is a homomorphism of Boolean algebras with added unary relation.)

In [2], the concept of an *open atomic* interior algebra was introduced, that is, an interior algebra A such that for all open elements $a > 0$ in A, there is an atom b of A with $b \leq a$. Similarly, we will call an interior algebra A *closed atomic* if and only if for all closed elements $a > 0$ in A, there is an atom b of A with $b \leq a$. Open atomicity and closed atomicity are independent of each other: Let B be the product of the two element Boolean algebra and the free Boolean algebra on $\aleph_0$ generators. Then B has a unique atom b. Let A $= \mathrm{Alg} \langle B, \{0,b,1\} \rangle$ and let C $= \mathrm{Alg} \langle B, \{0,b',1\} \rangle$. Then A is open atomic but not closed atomic while C is closed atomic but not open atomic. The concept of closed atomicity is needed in Corollary 2.7.

1. Bases and Subbases for Generalized Topologies

Definition 1.1: Let G be a generalized topology in a Boolean algebra B. A subset $S \subseteq G$ is called a *base* for G if and only if for all $a \in G$ there is an $X \subseteq S$ with $a = \Sigma X$. If A is an interior algebra, then a base for the generalized topology A^O is called a base for A. □

Lemma 1.2: Let A be an interior algebra and let S be a base for A. Then for all $a \in A$ we have $a^I = \Sigma \{b \in S : b \leq a\}$.

Proof: Let $a \in A$. Then $a^I = \max \{b \in A^O : b \leq a\}$. Thus a^I is an upper bound for $\{b \in S : b \leq a\}$. Let d be any upper bound for $\{b \in S : b \leq a\}$. Now $a^I \in A^O$ and so there is an $X \subseteq S$ with $a^I = \Sigma X$. Then for all $x \in X$ we have $x \leq a^I \leq a$ whence $x \leq d$. Thus $a^I \leq d$. It follows that $a^I = \Sigma \{b \in S : b \leq a\}$. □

Theorem 1.3: Let B be a Boolean algebra and let $S \subseteq B$. Then S is a base for a generalized topology in B if and only if the following hold:

(i) For all $a \in B$, $\Sigma \{b \in S : b \leq a\}$ exists.

(ii) If X is a finite subset of S then $\sqcap X = \Sigma \{b \in S : b \leq \sqcap X\}$.

Proof: Let S be a base for a generalized topology G in B. From Lemma 1.2 we see that (i) holds. Now suppose X is a finite subset of S. Then $\sqcap X \in G$ and so we see that $\Sigma \{b \in S : b \leq \sqcap X\} = \Sigma \{b \in G : b \leq \sqcap X\} = \sqcap X$ whence (ii) holds. Conversely suppose (i) and (ii) hold. Let G be the set of all joins of members of S. We show that G is a generalized topology in B. ϕ is a finite subset of S and $1 = \sqcap \phi$ whence $1 = \Sigma \{b \in S : b \leq 1\} \in G$. Let $a,b \in G$. Then c,d are joins of elements in S and so we in fact have $c = \Sigma \{b \in S : b \leq c\}$ and $d = \Sigma \{r \in S : r \leq d\}$. Then $cd = \Sigma \{cr : r \in S \text{ and } r \leq d\} = \Sigma \{\Sigma \{br : b \in S \text{ and } b \leq c\} : r \in S \text{ and } r \leq d\} = \Sigma \{br : b,r \in S, b \leq c \text{ and } r \leq d\}$. Thus $cd \in G$. Lastly consider $a \in B$. Put $c = \Sigma \{b \in S : b \leq a\}$. Then $c \in G$ and $c \leq a$. Let $d \in G$ with $d \leq a$. Then $d = \Sigma X$ for some $X \subseteq S$. Then for all $x \in X$, $x \leq d \leq a$ whence $x \leq c$. Thus $c = \max \{b \in G : b \leq a\}$. It follows that G is a generalized topology in B and so S is a base. □

Definition 1.4: Let G be a generalized topology in a Boolean algebra B. A subset $S \subseteq G$ is called a *subbase* for G if and only if the set $\{\sqcap X : X \text{ is a finite subset of } S\}$ is a base for G. If A is an interior algebra, then a subbase for the generalized topology A^O is called a subbase for A. □

Corollary 1.5: Let B be a complete Boolean algebra and let $S \subseteq B$.

(i) S is always a subbase for a generalized topology in B.

(ii) If $1 \in S$ and S is closed under binary meets then S is a base for a generalalized topology in B. □

Corollary 1.6: Let A be an interior algebra and let B be a complete Boolean algebra. If $f : \mathrm{Ba}\, A \longrightarrow B$ is an embedding then $f[A^O]$ is a base for a generalized topology G in B and moreover, $f : A \longrightarrow \mathrm{Alg} \langle B, G \rangle$ is a homomorphic embedding.

Proof: By Corollary 1.5 (ii), $f[A^O]$ is indeed a base for a generalized topology G in B. Let $a \in A$. By Lemma 1.2 we see that in $\mathrm{Alg} \langle B, G \rangle$, $f(a)^I = \Sigma \{f(b) : b \in A^O \text{ and } f(b) \leq f(a)\} = \Sigma \{f(b) : b \in A^O \text{ and } b \leq a\} = \Sigma \{f(b) : b \in A^O \text{ and } b \leq a^I\} = \Sigma \{f(b) : b \in A^O \text{ and } f(b) \leq f(a^I)\} = f(a^I)$ and the result follows. □

Remark 1.7: Corollary 1.6 is particularly important since it allows us to easily generalize the many results concerning complete extensions of Boolean algebras, to interior algebras. For example, every interior algebra is homomorphically embeddable in a complete interior algebra via an embedding that preserves any specified set of joins and meets. (cf. §4 of chap. 3 in [4].) □

Corollary 1.8: Let B be a complete Boolean algebra and let A be a complete interior algebra. If $f : B \longrightarrow$ Ba A is a complete surjective homomorphism then there is a generalized topology G in B with base $f^{-1}[A^{O}]$ such that $f : \mathrm{Alg}\,\langle B, G \rangle \longrightarrow A$ is a complete surjective homomorphism.

Proof: By Corollary 1.5 (i), $f^{-1}[A^{O}]$ is a base for a generalized topology G in B. Let $a \in B$. By Lemma 1.2 we see that in $\mathrm{Alg}\,\langle B, G \rangle$, $a^{I} = \Sigma\,\{b \leq a : f(b) \in A^{O}\}$. Thus $f(a^{I}) = \Sigma\,\{f(b) : f(b) \leq f(a) \text{ and } f(b) \in A^{O}\} = \Sigma\,\{c \leq f(a) : c \in A^{O}\} = f(a)^{I}$ and the result follows. □

2. Countability Properties for Interior Algebras

Definition 2.1: An interior algebra A is said to be *first countable* if and only if for each atom b in A, the filter N(b) is sequential. □

Definition 2.2: An interior algebra A is said to be *second countable* if and only if A has a countable base. □

Definition 2.3: An interior algebra A is said to be a *Lindelöf* algebra if and only if for all $S \subseteq A^{O}$, $\Sigma S = 1$ implies that $\Sigma X = 1$ for some countable $X \subseteq S$. □

Theorem 2.4: Any second countable interior algebra is first countable and Lindelöf.

Proof: Let A be second countable. Let R be a countable base for A. Consider an atom b in A. Put $U = \{r \in R : b \leq r\}$. Then $U \subseteq N(b)$. Let $d \in N(b)$. There is an $X \subseteq R$ with $d^{I} = \Sigma X$. Now $b \leq d^{I}$ and b is an atom so $b \leq r$ for some $r \in R$. Then $r \in U$ and $r \leq d$. Thus U is a

base for the filter N(a). Moreover U is countable since R is and so A is first countable. Now consider an $S \subseteq A^{O}$ with $\Sigma S = 1$. For all $a \in S$ there is an $R(a) \subseteq R$ with $a = \Sigma R(a)$. Put $V = \cup\,\{R(a) : a \in S\}$. For each $t \in V$ there is an $a(t) \in S$ with $t \in R(a(t))$. Put $W = \{a(t) : t \in V\}$. Then $|W| \leq |V| \leq |R|$ and so $W \subseteq S$ is countable. Let d be an upper bound for W. Then for all $t \in V$, $t \leq \Sigma R(\, a(t)\,) = a(t) \leq d$. Thus d is an upper bound for V. Hence for all $a \in S$, d is an upper bound for R(a) whence $a \leq \Sigma R(a) \leq d$. Thus d is an upper bound for S and so $d = 1$ whence $\Sigma W = 1$. Thus A is Lindelöf. □

Remark 2.5: Let X be a topological space. Then we see that $\mathcal{P}(X)$ is first countable, second countable or Lindelöf as an interior algebra if and only if X is first countable, second countable or Lindelöf respectively, as a topological space. A first countable topological space need not be Lindelöf and vice versa. A first countable Lindelöf topological space need not be second countable. (See [5].) Thus we see that the converse of Theorem 2.4 fails and the classes of first countable and Lindelöf interior algebras are incomparable even when we restrict our attention to complete atomic interior algebras. □

Proposition 2.6:

Let A be an interior algebra. Consider the following assertions:

(i) A is a Lindelöf algebra.

(ii) Every countably complete proper filter in $A^{\square}$ has a non–zero lower bound in $A^{\square}$.

(iii) Every countably complete proper filter in A has a non–zero accumulant.

We always have (i) ⟹ (ii) ⟹ (iii). If A is countably complete then (i), (ii) and (iii) are all equivalent.

Proof: Assume (i). Let F be a countably complete proper filter in $A^{\square}$. Suppose $\sqcap F = 0$. By duality and the definition of Lindelöf there is a countable $S \subseteq F$ with $\sqcap S = 0$. (Recall that meets in $A^{\square}$ and A coincide.) But then $0 \in F$, a, contradiction. Hence there is a lower bound $b > 0$ for F in $A^{\square}$. Thus (i) ⟹ (ii). Assume (ii). Let F be a countably complete proper filter in A. Then $F \cap A^{\square}$ is a countably complete proper filter in $A^{\square}$. Let b be a non–zero lower bound for $F \cap A^{\square}$. Noting that $F \cap A^{\square} = \{a^C : a \in F\}$ we see that $F \longleftarrow b$. Thus (ii) $\Longrightarrow$ (iii). Now suppose that A is countably complete. Assume (iii). Let $S \subseteq A^{\square}$ with $\sqcap S = 0$. Suppose that $\sqcap R > 0$ for all countable $R \subseteq S$. Let $F = \{b \in A : b \geq \sqcap R$ for some countable $R \subseteq S\}$. Clearly F is a proper filter. Let W be a countable subset of F. For all $a \in W$ there is a countable subset $R(a) \subseteq S$ with $a \geq \sqcap R(a)$. Put $R = \cup \{R(a) : a \in W\}$. Then R is a countable subset of S. Then for all $a \in W$, $a \geq \sqcap R(a) \geq \sqcap R$ and so $\sqcap W \geq \sqcap R$ whence $\sqcap W \in F$. Thus F is countably complete. Hence $F \longleftarrow b$ for some $b > 0$. But $S \subseteq F$ and so for all $a \in S$, $b \leq a^C = a$, a contradiction since $\sqcap S = 0$. Thus $\sqcap R = 0$ for some countable $R \subseteq S$. By duality we see that A is Lindelöf. Thus (iii) $\Longrightarrow$ (i) and so it follows that (i), (ii) and (iii) are all equivalent. □

Theorem 2.7: Let A be a countably complete interior algebra. The following are equivalent:

(i) Every countably complete proper filter in A accumulates at an atom.

(ii) A is a closed atomic Lindelöf algebra.

Proof: Assume (i). By Proposition 2.6, A is Lindelöf. Let $b > 0$ be closed in A. Then $[b)$ is a countably complete (in fact complete) proper filter in A and so $[b) \longleftarrow a$ for some atom a of A. But then $a \leq b^C = b$. Therefore A is closed atomic. Thus (i) $\Longrightarrow$ (ii). Now assume (ii). Let F be a countably complete proper filter in A. Since A is Lindelöf, $F \longleftarrow b$ for some $b > 0$. Then $F \longleftarrow b^C$. Also $b^C > 0$ and so there is an atom $a \leq b^C$ whence $F \longleftarrow a$. Thus (ii) $\Longrightarrow$ (i). □

Remark 2.8: Let A be an interior algebra. Suppose $c \in A$ and b is an atom in A. If there is a non–zero sequence (a_n) in $(c]$ with $(a_n) \longrightarrow b$ then by Proposition 4.21 of [2] we see that $b \ll c$. However the converse fails even for complete atomic interior algebras: Let X be the co–countable space on $\aleph_1$. (See [5].) Then in X we have $\{0\}'^C = \aleph_1$ and so $0 \in \{0\}'^C$. But there is no sequence (x_n) in $Z = \{0\}'$ with $(x_n) \longleftarrow 0$ in X. For suppose (x_n) is such a

sequence. Then $Y = \aleph_1 - \{x_n : n < \omega\}$ is a neighbourhood of 0 in X and so for some $k < \omega$, $x_k \in Y$, a contradiction. By Remark 4.23 of [2] we see that there is no non–zero sequence (B_n) in the ideal $(Z]$ of $\mathcal{P}(X)$, such that $(B_n) \longleftarrow \{0\}$. However, $\{0\} \leq \{0\}'^C$ in $\mathcal{P}(X)$. □

The above remark motivates the following definition:

Definition 2.9: An interior algebra A is said to be *sequentially determined* if and only if for all $c \in A$ and all atoms b in A, $b \ll c$ implies that $(a_n) \longrightarrow b$ for some non–zero sequence (a_n) in $(c]$. □

The following theorem follows directly from Proposition 4.21 of [2].

Theorem 2.10: Any first countable interior algebra is sequentially determined. □

Remark 2.11: The converse of Theorem 2.10 fails even for complete atomic interior algebras: Let X be the co–finite space on $\aleph_1$. (See [5].) Suppose that the neighbourhood filter of 0 has a countable base with enumeration $\{B_n : n < \omega\}$. $C = \cap \{B_n : n < \omega\}$ is a countable intersection of co–finite sets and so it is co–countable. In particular there is an $x \in C - \{0\}$. Then $\{x\}'$ is a neighbourhood of 0 and so $B_n \subseteq \{x\}'$ for some $n < \omega$. Then $x \notin B_n$, a contradiction. Thus X is not a first countable space. However $\mathcal{P}(X)$ is sequentially determined: Let $z \in \aleph_1$ and suppose $Y \subseteq \aleph_1$ with $z \in Y^C$ in X. If Y is finite then $z \in Y$ and so putting $x_n = z$ for for all $n < \omega$ gives a sequence (x_n) in Y with $(x_n) \longrightarrow z$ in X. If Y is infinite let (x_n) be any injective sequence in Y. Let U be an open neighbourhood of z. Suppose that for all $k < \omega$ there is an $m(k) \geq k$ with $x_{m(k)} \notin U$. Then $\{x_{m(n)} : n < \omega\}$ is an infinite subset of U', a contradiction since U is co–finite. Thus there is a $k < \omega$ with $x_n \in U$ for all $n \geq k$. Thus $(x_n) \longrightarrow z$ in X. It follows that $\mathcal{P}(X)$ is sequentially determined. □

3. Preservation of Countability Properties under Principal Quotients

Lemma 3.1: Let $f : A \longrightarrow C$ be a topomorphism between two interior algberas. Then for all $a \in A$, $f[N(a)] \subseteq N(f(a))$. (Topomorphisms preserve neighbourhoods.) □

Lemma 3.2: Let A be an interior algebra and let $c \in A$. Then for all $b \in (c]$, $N^c(b) = \{ac : a \in N(b)\}$.

Proof: Let $b \in (c]$. The map $a \longmapsto ac$ is the canonical topomorphism from A onto (c]. (See [5].) Thus by Lemma 3.1 we see that if $a \in N(b)$, then $ac \in N^c(b)$. Conversely consider a neighbourhood d of b in (c]. There is an open element e of (c] with $b \leq e \leq d$. Then there is an open element v of A with $e = vc$. Put $a = v + d$. Then $a \geq v \geq e \geq b$ and so a is a neighbourhood of b in A. Moreover $ac = vc + dc = e + d = d$, and the result follows. □

Theorem 3.3: The following classes of interior algebras are closed under principal quotients:

(i) First countable interior algebras.

(ii) Second countable interior algebras.

(iii) Sequentially determined interior algebras.

Proof: (i): Let A be first countable and let $c \in A$. Let b be an atom in (c]. Then b is an atom in A and so there is a countable base B for the filter N(b) in A. But by Lemma 3.2 the neighbourhood filter of b in (c] is just $N^c(b) = \{ac : a \in N(b)\}$. Now $C = \{ac : a \in B\}$ is a countable subset of $N^c(b)$ and for all $a \in N(b)$ there is an $e \in B$ with $e \leq a$ whence $ec \in C$ and $ec \leq ac$. Thus C is a countable base for $N^c(b)$ and the result follows. (ii): Let A be a second countable interior algebra and let $c \in A$. There is a countable base S for A. Put $R = \{ac : a \in S\}$. Then R is a countable subset of $(c]^O$. Let $d \in (c]^O$. There is a $b \in A^O$ with $d = bc$. Then there is an $X \subseteq S$ with $\Sigma X = b$. But then $d = \Sigma \{ac : a \in X\}$ and $\{ac : a \in X\} \subseteq R$. Thus R is a base for (c] and the result follows. (iii): Let A be a sequentially determined interior algebra and let $c \in A$. Let b be an atom in (c] and let $b \ll d$ in (c]. Then $b \leq d^{C,c} = cd^C \leq d^C$ and so $b \ll d$ in A. Hence there is a non-zero sequence (a_n) in (d] with $(a_n) \longrightarrow b$ in A. Let e be a neighbourhood of b in (c]. By Lemma 3.2 there is a neighbourhood v of b in A with $e = vc$. Hence there is a $k < \omega$ with $a_n \leq v$ for all $n \geq k$. But for all $n < \omega$, $a_n \leq d \leq c$ and so for all $n \geq k$, $a \leq vc = e$. Thus $(a_n) \longrightarrow b$ in (c] and the result follows. □

Theorem 3.3 (ii) leads to the following theorem characterizing second countable interior algebras:

Theorem 3.4: The following are equivalent for an interior algebra A:

(i) A is second countable.

(ii) Any base for A contains a countable base for A.

Proof: Clearly (ii) ⟹ (i) since A^O is a base for A. Assume (i). Then there is a countable base R for A. Suppose S is a base for A. Let $b \in R$. There is an $X \subseteq S$ with $\Sigma X = b$. Then $X \subseteq (b]$ and so in fact $X \subseteq (b]^O$ and $\Sigma X = b$ in (b]. By Theorem 3.3 (ii), (b] is second countable. By Theorem 2.4, (b] is Lindelöf. Hence there is a countable $T(b) \subseteq X$ with $\Sigma T(b) = b$ in (b] and hence in A. Put $T = \cup \{T(b) : b \in R\}$. Then T is a countable subset of S. Consider $d \in A^O$. There is a $W \subseteq R$ with $\Sigma W = d$. Then $d = \Sigma \{\Sigma T(b) : b \in W\} = \Sigma (\cup \{T(b) : b \in W\})$ and moreover $\cup \{T(b) : b \in W\} \subseteq T$. Therefore T is a base for A. Thus (i) ⟹ (ii). □

Theorem 3.5: The class of Lindelöf algebras is closed under closed quotients.

Proof: Let A be a Lindelöf algebra and let c be closed in A. Let $R \subseteq (c]^O$ with $\Sigma R = c$ in (c] and hence in A. Then for all $r \in R$ there is a $b(r) \in A^O$ with $r = cb(r)$. Suppose d is an upper bound for $\{b(r) : r \in R\} \cup \{c'\}$. Then for all $r \in R$, $r = cb(r) \leq cd$. Hence $c = cd$, that is $c \leq d$. But $c' \leq d$ and so $d = 1$. Thus $\Sigma (\{b(r) : r \in R\} \cup \{c'\}) = 1$. Hence there is a countable $W \subseteq R$ with $\Sigma (\{b(r) : r \in W\} \cup \{c'\}) = 1$. But then $c = \Sigma (\{cb(r) : r \in W\} \cup \{cc'\}) = \Sigma (W \cup \{0\}) = \Sigma W$. Thus (c] is a Lindelöf algebra. □

Remark 3.6: Let X be a topological space and let A be the power set interior algebra determined by X. If $Y \subseteq X$ then the ideal algebra $(Y]$ of A is the power set interior algebra determined by the subspace Y of X with underlying set Y. We thus see that since an open subspace of a Lindelöf space need not be Lindelöf (see [5]), an open quotient of a (complete atomic) Lindelöf algebra need not be Lindelöf. □

The above remark motivates the following definition:

Definition 3.7: An interior algebra A is called an *hereditarily Lindelöf* algebra if and only if every principal quotient of A is a Lindelöf algebra. □

Remark 3.8: Since any interior algebra is a principal quotient of itself, an hereditarily Lindelöf algebra is Lindelöf. The class of hereditarily Lindelöf algebras is also trivially closed under principal quotients. □

Remark 3.9: By the correspondence between subspaces and ideal algebras we see that if X is a topological space then $\mathcal{P}(X)$ is an hereditarily Lindelöf algebra if and only if X is hereditarily Lindelöf according to the usual sense in topology, i.e. every subspace of X is Lindelöf. □

By Theorem 2.4 and Theorem 3.3 (ii) we have:

Corollary 3.10: Every second countable interior algebra is hereditarily Lindelöf. □

Remark 3.11: The converse of the Corollary 3.10 fails. Let X be the co–finite space on $\aleph_1$. In Remark 2.11 we saw that X is not first countable and hence not second countable. However any subspace of X is a co–finite space which is compact and hence Lindelöf. Thus

$\mathcal{P}(X)$ is an hereditarily Lindelöf algebra. This follows from the fact that if Y is a subspace of X, then $\mathcal{P}(Y)$ is isomorphic to the ideal algebra $(Y]$ of $\mathcal{P}(X)$. □

4. Preservation of Countability Properties under Joins of Interior Algebras

Definition 4.1: Let A be an interior algebra and let $\{A_i : i \in I\}$ be a family of interior algebras. Suppose that there are elements $\{a_i : i \in I\}$ in A such that $\Sigma_I a_i = 1$ and $A_i \cong (a_i]$ for all $i \in I$. Then we say that A is a *join* of the interior algebras $\{A_i : i \in I\}$. We may refer to a join as being *open*, *closed* or *clopen* etc. if we can choose $\{a_i : i \in I\}$ to be a set of open, closed or clopen elements etc. respectively. □

Remark 4.2: Let $\{A_i : i \in I\}$ be a family of interior algebras. Then the product $\Pi_I A_i$ is a clopen join of the family $\{A_i : i \in I\}$ since for all $j \in I$, $A_j \cong (a]$ where $a = (a_i)$ is given by $a_j = 1$ and $a_i = 0$ for $i \neq j$. □

Remark 4.3: Let X be a topological space and let $\{X_i : i \in I\}$ be a family of subspaces of X such that $X = \cup_I X_i$. Recalling that subspaces correspond to ideal algebras we see that $\mathcal{P}(X)$ is a join of the interior algebras $\{\mathcal{P}(X_i) : i \in I\}$. □

Definition 4.4: A topomorphism $f : A \longrightarrow B$ between two interior algebras is called a *principal quotient topomorphism* if and only if ker(f) is a principal congruence on the generalized topological space Gt A. □

Proposition 4.5: Let A be an interior algebra and let $\{A_i : i \in I\}$ be a family of interior algebras. Then A is a join of $\{A_i : i \in I\}$ if and only if Ba A is a subdirect product of $\{\text{Ba } A_i : i \in I\}$ and for all $i \in I$, the ith projection $\pi_i : \text{Ba } A \longrightarrow \text{Ba } A_i$ is a principal quotient topomorphism $\pi_i : A \longrightarrow A_i$ when considered to be a map from A to A_i. □

The above follows easily from the correspondence between ideal algebras and principal quotients of interior algebras. (See Theorem 3.7 of [3].)

Theorem 4.6: The following classes of interior algebras are closed under open joins.

(i) First countable interior algebras.

(ii) Sequentially determined interior algebras.

Proof: (i): Let A be an interior algebra and let $S \subseteq A^O$ with $\Sigma S = 1$ and (a] first countable for all $a \in S$. Let b be an atom in A. Then $b \leq a$ for some $a \in S$. There is a countable base B for the neighbourhood filter $N^a(b)$ of b in (a]. Note that since a is open $N^a(b) = N(b) \cap (a]$ and so in fact $B \subseteq N(b)$. Let $d \in N(b)$. Then $ad \in N^a(b)$ and so there is a $v \in W$ with $v \leq ad \leq d$. Thus B is a countable base for N(b) and the result follows. (ii): Let A be an interior algebra and $S \subseteq A^O$ with $\Sigma S = 1$ and (a] sequentially determined for all $a \in S$. Let b be an atom in A and let $c \in A$ with $b \ll c$. Then there is a $d \in S$ with $b \leq d$. Note that since d is open the quotient map from A onto (d] given by $r \longmapsto dr$ for all $r \in A$, is a homomorphism and so we see that $b \ll dc$ in (d]. Hence there is a non–zero sequence (a_n) in (dc] with $(a_n) \longrightarrow b$ in (d]. But then (a_n) is a non–zero sequence in (c] and using Lemma 3.2 we see that $(a_n) \longrightarrow b$ in A whence the result follows. □

Remark 4.7: The classes of first countable and sequentially determined interior algebras are not closed under closed joins: Let X be the co–countable space on $\aleph_1$. In Remark 2.8 we saw that $\mathcal{P}(X)$ was not sequentially determined and hence also not first countable. However, every point of X is closed and so it follows by Remark 4.3 that $\mathcal{P}(X)$ is a closed join of $\aleph_1$ copies of the unique two element interior algebra which is trivially first countable and hence also sequentially determined.

Remark 4.8: If X is the topological sum of a family of topological spaces $\{X_i : i \in i\}$, then $\mathcal{P}(X) \cong \Pi_I \mathcal{P}(X_i)$. (An isomorphism from $\Pi_I \mathcal{P}(X_i)$ to $\mathcal{P}(X)$ is given explicitly by the mapping $(S_i) \longmapsto \bigcup_I (S_i \times \{i\})$ where for all $i \in I$, $S_i \subseteq X_i$.) Since the classes of Lindelöf, hereditarily Lindelöf, and second countable topological spaces are not closed under topological sums (see [5]) it follows that the classes of Lindelöf, hereditarily Lindelöf, and second countable interior algberas are not closed under products let alone arbitrary joins. □

Despite the negative result of the above remark we have the following results:

Theorem 4.9: The following classes are closed under countable joins (in particular, under countable products):

(i) Lindelöf algebras.

(ii) Hereditarily Lindelöf algebras.

Proof: (i): Let A be an interior algebra and let $S \subseteq A$ with $\Sigma S = 1$ and (a] Lindelöf for all $a \in S$. Let $R \subseteq A^O$ with $\Sigma R = 1$. Then for all $a \in S$, $\Sigma \{ar : r \in R\} = a$ in A hence in (a]. But $\{ar : r \in R\} \subseteq (a]^O$ and so there is a countable subset $T(a) \subseteq R$ with $\Sigma \{ar : r \in T(a)\} = a$ in (a] hence in A. Put $T = \bigcup \{T(a) : a \in S\}$. Then T is a countable subset of R. Suppose d is an upper bound of T. Then for all $a \in S$, d is a upper bound of T(a) and hence of $\{ar : r \in T(a)\}$. Thus for all $a \in S$, $d \geq \Sigma \{ar : r \in T(a)\} = a$. Thus $d = 1$ and so $\Sigma T = 1$. Thus A is a Lindelöf algebra. (ii): Let A be an interior algebra and let $S \subseteq A$ with $\Sigma S = 1$ and (a] hereditarily Lindelöf for all $a \in S$. Let $b \in A$. Then $b = \Sigma \{ab : a \in S\}$ in A and hence in (b]. Also for all $a \in S$, (ab] is a Lindelöf algebra since it is a principal quotient of (a]. Thus (b] is a join of Lindelöf algebras and so it is Lindelöf by (i). Thus A is hereditarily Lindelöf. □

Theorem 4.10: The class of second countable interior algebras is closed under countable open joins (in particular, under countable products).

Proof: Let A be an interior algebra and let $S \subseteq A^O$ be countable with $\Sigma S = 1$ and let (a] be second countable for all $a \in S$. Then for all $a \in S$ there is a countable base R(a) for (a]. Moreover since $a \in A^O$ we have $R(a) \subseteq A^O$. Putting $R = \bigcup \{R(a) : a \in S\}$ gives a countable subset of A^O. Now if $a \in S$ and $b \in (a]^O$ there is an $T(a,b) \subseteq R(a)$ with $b = \Sigma\, T(a,b)$ in $(a]$ and hence in A. Now if $d \in A^O$, $ad \in (a]^O$ for all $a \in S$ and so we have $d = \Sigma\,\{ad : a \in S\} = \Sigma\,\{\Sigma\, T(a,ad) : a \in S\} = \Sigma\, T(d)$ where $T(d) = \cup\,\{T(a,ad) : a \in S\}$. Then for all $d \in A^O$, $T(d) \subseteq R$ and so R is a base for A whence the result follows. □

Remark 4.11: The class of second countable interior algebras is not closed under countable closed joins: Let X be the Arens–Fort space. (See [5] for details.) X is not second countable but it is T_1 (whence every point is closed) and countable [5]. Thus by Remark 4.3, $\mathcal{T}(X)$ is a countable closed join of $\aleph_0$ copies of the two element interior algebra which is trivially second countable. □

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